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A Review of Analytical Methods for Calculating Static Pressures in Bulk Solids Storage Structures †

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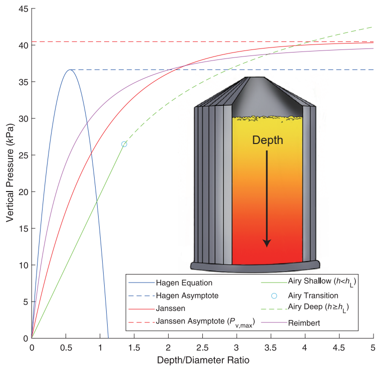
The Janssen equation is a widely used method for calculating pressures in bulk storage structures. This review explores the historical legacy of Janssen's equation and its applications in both planar and three-dimensional structures. Our focus is on the limitations of the original formulation of Janssen, extensions made to avoid these deficiencies, and alternative models that have been developed. The motivation behind these modifications is to improve the representation of shear stress within a grain bin in both the horizontal and vertical directions. Modifications to Janssen's basic assumptions include the vertical-to-horizontal stress ratio (k), the coefficient of friction between the wall and the stored bulk material (μ), internal angle of friction (ϕ), and bulk density (ρ). We also discuss recent developments in pressure theories, which have provided new insights into pressure fields in bulk storage bins. These modern approaches include the continuum elastic theory and microscopic theory. Finally, we discuss recent developments in pressure theories which provide new insights into the storage of bulk solids. Overall, this review provides a comprehensive overview of the Janssen equation and its historical development, limitations, and extensions, as well as recent advancements in pressure theory that offer a more accurate representation of pressure fields in bulk storage structures.

Keywords: powder, particle, bulk solids, Janssen effect, stress distribution, granular matter



1. Introduction

A bulk solid is an aggregate material consisting of solid particles and fluids (gases/air). Considered as a whole, the material has properties common to both fluids and solids, i.e., it can change its shape to fill volumes like a liquid but responds to internal shear stresses like a solid (Walker, 1966). Bulk solids occupy an intermediate position on the spectrum between liquids and solids. These materials exhibit many novel, counter-intuitive properties, for example the stress (pressure) distribution within the material when it is stored in bins. This behavior is associated with "the Janssen effect", as pressure is commonly described by H.A. Janssen's equation. The Janssen effect predicted that the bottom of a bin did not experience the full force of the weight of the commodity stored below some critical height. This phenomenon resulted from the friction between the sidewall and the stored material: the walls carried some of the material weight. Knowledge of internal and wall friction is essential to Janssen's theory. Coulomb's work on friction and failure of soils due to shearing led to the first understanding of the relationship between internal stress and the orientation of failure planes within a material. Rankine synthesized Coulomb's work and Mohr's description of failure planes and provided an analysis of material failure. This paper will first discuss the historical developments leading up to pressure models before discussing Janssen's work, and then investigate important refinements and critiques of the model over the past century. There are three modern works with a similar historical perspective on the topic of pressures in bulk storage structures. The first was the translation of Janssen's entire original paper into English, 111 years after its original publication and a century's worth of citations (Sperl, 2006). This paper contains original data and figures as well as a short commentary that speaks to the legacy of the original publication. The second and third publications were conference papers by A.W. Roberts presented during the PARTEC Congress held in Nuremberg, Germany, 21–23 March 1995. He provided a detailed description of Janssen's work, and a review of the past 100 years of bulk solids research (1895–1995; Roberts, 1995). The work was formally rewritten and later published (Roberts, 1998). The historical section is indebted to Milo Ketchum's "The Design of Bins, Walls and Grain Elevators" (Ketchum, 1911). While this paper provided a novel interpretation of Janssen's equation, the alternative models have not been as resilient to time as Janssen's solution. Furthermore, the paper was written in 1998, and advancements in computational technologies have allowed for new

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approaches to modeling bulk solids, which were before "insoluble problems" as noted by Jenike (1961). There will be a brief overview of the experiments covered in this literature review for historical context. Information about the pressures in a bin can also provide insight into a wide range of physical phenomena. The pressure field affects bulk solid's packing, flowability, porosity, and tortuosity, to name a few. For example, Janssen along with empirical packing equations has been used to predict the density change within a bin (Thompson et al., 1991; Bhadra et al., 2015, 2018; Turner et al., 2016; Cheng et al., 2017).

2. Historical Pressure Theories

This section focuses on a review of pressure models for grain storage bins. Historically, the "Janssen Effect" was known before Janssen's mathematical description, noted anonymously by Gotthilf Heinrich Ludwig Hagen. Hagen's account is the oldest published theory of pressure in bulk materials, 40 years before Janssen in 1852 (Hagen, 1852). Hagen attempted to predict saturation pressure with respect to height (sand). The paper was translated into English by Tighe et al., which provided a detailed view of this seminal work (Tighe and Sperl, 2007). Hagen suggested that the friction on the side of the wall acted in the upward direction, opposing to the gravitational force. The pressure felt by a "disk which is easily movable but seals tightly" was given by the expression,

$$\sigma_{\text{disc}} = r^2 \pi \rho g y - 2 r \rho g \mu y^2 \tag{1}$$

where r is the radius of the disc (bin), ρ the bulk density, y the height of bulk material, and μ the friction coefficient. This equation reached a maximum at a characteristic height—related to the diameter of the bin, height, and friction coefficient—and began to decrease afterwards, a mathematical prediction but physically incorrect; Hagen stated that the pressure followed this curve until this maximum was reached. After, the bin pressure remained constant at this maximum, which was a similar behavior to the "Janssen Effect". The difference from Janssen's formulation was that Hagen used a quadratic to explain the change in pressure, whereas Janssen's approach solved a differential equation whose solution was an exponential function, which allowed for an asymptotic approach to the maximum pressure. A noteworthy reference found in Hagen's work regarded an observation Huber-Bernand published in 1829, likely the first published research on the so-called "Janssen Effect," which was highlighted in both Hagen and Janssen's publications. Huber-Bernand noticed that when he filled a container with eggs, added sand in the voids between the eggs extending several inches above the eggs, as well as a 25 kg weight on top, the eggs placed at the bottom of the container remained unbroken, even though all this weight should crush them. He concluded that only a portion of the pressure was transmitted to the bottom of the container, indicating some kind of saturation depth (Sperl, 2006), (Tighe and Sperl, 2007). A Welsh engineer Isaac Roberts published a crucial pre-Janssen study that observed an asymptotic trend in bulk solid bin pressures. He claimed that the pressures at the bottom of bins stopped increasing at the height of two diameters, which was an empirical demonstration of the Janssen effect (Roberts, 1883). He conducted two experiments on this topic. The first was prone to measurement errors due to the mechanical nature of the scales used, an issue other researchers (e.g. Prante) would struggle with and overcome; the second experiment produced reliable results that were widely referenced (Ketchum, 1911). Ten years later, H.A. Janssen formulated his seminal equation analytically modeling Roberts's work, although the work was not cited. Both Hagen and Roberts worked on mathematical descriptions of stresses of earth pressure systems and experimental data from agricultural commodities. H.A. Janssen would be the first to formulate a simple way to calculate pressures in deep bins, the standard reference for well over a century on the topic. This equation's influence was enormous and has been used in almost all design codes and standards in the world, such as the European bulk storage building codes (European Standard EN 1991-4, 2006). The expression was also used in the calculation of apparent weight for bin capacities by the ASABE (American Society of Agricultural and Biological Engineers; ASABE, 2005). Janssen's work has been used by a vast community of researchers and industry because it explained the complicated phenomenon of bin pressures via a straightforward expression (**Eqn. (7)**). Janssen was an engineer from Germany; not much is known about the individual and his work outside of this one publication, a seminal paper for the science of bulk solids. Roberts postulates that it was likely his work and records were destroyed during the Second World War (Roberts, 1995). The life and work of Janssen is accessed through his 1895 paper, "Experiments on Corn Pressure in Silo Cells." One year after its publication, an English translation of Janssen's abstract was published in the Proceedings of the Institution of Civil Engineers, the work often cited as Janssen (1896). Soon, the equation was added to the building practices outline in the "Des Ingenieurs Taschenbuch or The Hütte - Das Ingenieurwissen", a book of references for practicing engineers, initially written in German and now translated into many languages (Ketchum, 1911). During this time, many aspects of the work were validated, critiqued, and modified, which is discussed later in this review. It should also be noted that M. Koenen worked on a similar theory of pressure but published it a year after Janssen. Koenen's work is challenging to find because it was never translated from German to English, and the conclusions were passed down by secondary literature. His most significant contribution suggested the lateral to vertical pressure ratio could be calculated using established equations, such as Rankine's active pressure state (**Eqn. (14)**), while Janssen experimentally determined its value (Reimbert M. and Reimbert A., 1976; Schulze, 2008; Koenen, 1896). Koenen explicitly connected Janssen's work to the Mohr-Coloulmb failure criterion by relating it to the principal stresses acting on a mass. It is an interesting historical point that Janssen referenced two other works incorrectly, which Sperl addressed in the translation of Janssen's original document (Sperl, 2006). The first was a fellow German researcher, C. Arndt, and was used to discuss the then-new developments in grain storage in North America. The paper's title is "The Silos of Galati and Braila", which are Romanian cities that hold the oldest concrete grain bins. These were built in the late 1880s, designed by an engineer Anghel Saligny, and were two of the earliest examples of reinforced concrete bins. Janssen described these bins as "...iron-strengthened brickwork, and in six-fold profiles like honeycombs. The grain was introduced through the upper end of the cell through a hatch. For discharge, stock transfer, or embarkation, the bottoms of the cells provided close-able openings" (Sperl, 2006). This sounds similar to modern grain storage bin design; Janssen saw these as the future of bin design and wanted to provide mathematical models for predicting pressures. The second citation was Hagen, who has been discussed thoroughly above (Sperl, 2006). Ketchum discussed the design of concrete bins in Chapter XVIII of his text "The Design of Bins, Walls and Grain Elevators". Unfortunately, he did not mention Saligny and only discussed the concrete bins of North America.

2.1 Quantitative description of the Janssen effect

Janssen began with a force-balance equation of a thin horizontal layer of bulk material in a deep bin. Note two things: first, Janssen was investigating deep bins but the theory can be applied to shallow bins (Xu and Liang, 2022). Second, the following is not directly Janssen's formulation but a modernization of the derivation for clarity that was made popular by Ketchum, but is essentially the same (Ketchum, 1911). The first difference lies in incorporating a hydraulic radius R_H, which generalized the container geometry from Cartesian (rectangles) to arbitrary shapes. Secondly, in Janssen's work, he expressed $K = \mu k$ and solved for K experimentally. The lateral pressure ratio, k, is defined as the ratio of the horizontal stress to the vertical effective stress. Within a filled container (bin), the force balance on an infinitesimal thin disk element of granular material in the vertical direction is given as:

$$A\sigma_y + g\rho A\,\mathrm{d}y = A\left(\sigma_y + \mathrm{d}\sigma_y\right) + \tau_\mathrm{w} C\,\mathrm{d}y \tag{2}$$

where A is the cross-sectional area and C the circumference of the bin, ρ the bulk density of the material, g the gravitational acceleration, σ_x and σ_y the horizontal and vertical stresses, respectively, and τ_w the shear stress acting on the material at the wall, which opposes the gravitational force.

Here, x and y are the horizontal and vertical coordinates, respectively. It should be noted that the depth of the stored material is given by y, increasing in the downward direction. The shear stress is related to the normal stress in the horizontal direction through the wall friction coefficient μ as follows:

$$\tau_\mathrm{w} = \sigma_x \mu = \sigma_x \tan\phi_\mathrm{w} \tag{3}$$

with $\mu = \tan\phi_\mathrm{w}$ given by the wall friction angle ϕ_w. Finally, the ratio of horizontal (σ_x) to vertical stress (σ_y) is assumed to be constant,

$$k = \frac{\sigma_x}{\sigma_y} \tag{4}$$

Defining the hydraulic radius

$$\frac{A}{C} = R_\mathrm{H} \tag{5}$$

substituting **Eqns. (3)** and **(4)** into **Eqn. (2)** yields the Janssen differential equation,

$$\frac{\mathrm{d}\sigma_y}{\mathrm{d}y} + \frac{\mu k}{R_\mathrm{H}}\sigma_y = g\rho \tag{6}$$

Solving for vertical pressure gives

$$\sigma_y = \frac{\rho g R_\mathrm{H}}{\mu k}\left(1 - \mathrm{e}^{-\frac{\mu y k}{R_\mathrm{H}}}\right) \tag{7}$$

Janssen's equation may also be derived from the fundamental (standard) static equilibrium equations for an infinitesimal element of material in $2D$ Cartesian or cylindrical coordinates. For a $2D$ Cartesian coordinate system, the equilibrium equations on an infinitesimal element of material in the vertical and horizontal directions are given as:

$$\begin{aligned} \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} &= g\rho \\ \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \end{aligned} \tag{8}$$

For the axisymmetric conditions where a cylindrical coordinate system (r,θ,z) is used, the following equilibrium equations are used:

$$\begin{aligned} \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + \frac{\partial\tau_{rz}}{\partial r} &= g\rho \\ \frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} &= 0 \end{aligned} \tag{9}$$

These formulae were used by Rahmoun et al. (2008) and Schillinger & Malla (2008) among others to derive pressure equations. The equilibrium approach is well-established in the field of continuum mechanics (Negi and Ogilvie, 1977), and Janssen's results can be obtained from these expressions, with certain assumptions. For example, assuming the shear stress is related to the normal stress by equation $\tau_{rz} = \mu\sigma_r$ and using the ratio of principal stresses, $\sigma_r/\sigma_z = k$, we obtain

$$\tau_{rz}(z) = \mu k \sigma_z(z) \tag{10}$$

Combining **Eqns. (9)** and **(10)** with the definition of the hydraulic radius $R_H = b/2$ reproduces the Janssen's differential **Eqn. (6)**.

2.2 Lateral pressure ratio *k*

In addition to producing the equation that bears his name, Janssen also gathered experimental data to validate his model for calculating pressures in bulk storage structures (**Eqn. (7)**). With novel instruments and measuring techniques, Janssen measured μ, the pressure exerted on the bottom of the bin (σ_y) and had planned to measure the horizontal force on the wall (σ_x) but failed to succeed due to the arching effects of the material and the construction of the apparatus. He suggested an improvement that was adapted by Jamieson in 1906. To solve **Eqn. (6)**, Janssen assumed that k was a constant, an assumption that many would question (Ketchum, 1911). Typically, the vertical-to-horizontal stress ratio k is assumed—unrealistically—to be constant throughout a bin, in the range 0 for rigid solids and 1 for liquids. For granular materials, the Rankine coefficient is often used which gives the stress ratio in terms of the internal angle of friction ϕ,

$$k = \frac{1-\sin\phi}{1+\sin\phi} \tag{11}$$

In fact, even with the assumption of constant k there is still a wide variety of values cited throughout the literature. This lack of consensus exists because k is not a fundamental, innate physical property of bulk solids; it is an emergent result of equilibrium, arising from the "fluid" nature of bulk materials. The value of k is not realistically expected to be constant within a storage structure at all. The simplifying assumption is made because Janssen's equation works well within a narrow column of material, but the variation of k through a large volume of bulk material has not been well-studied experimentally. It has been difficult to include a varying k analytically. Coulomb, Rankine and Janssen all made the simplifying assumption of constant $0 < k < 1$. Coulomb's approach produced a stress ratio which related lateral and vertical pressure of soil, and was determined by using the Moore-Coulomb failure criterion (Labuz and Zang, 2012) as follows:

$$k_a = \frac{\cos^2(\phi-\omega)}{\cos^2(\omega)\sin(\phi_w+\beta)\left(1+\sqrt{\dfrac{\sin(\phi_w+\phi)\sin(\phi-\beta)}{\cos(\phi_w+\omega)\cos(\omega-\beta)}}\right)^2} \tag{12}$$

$$k_p = \frac{\cos^2(\phi+\omega)}{\cos^2(\omega)\sin(\phi_w-\omega)\left(1-\sqrt{\dfrac{\sin(\phi_w+\phi)\sin(\phi+\beta)}{\cos(\phi_w-\omega)\cos(\beta-\omega)}}\right)^2} \tag{13}$$

where k_a is the active pressure ratio (the bulk solid exerts pressure on the wall), k_p the passive pressure ratio (the wall exerts pressure on the bulk solid), ϕ_w the friction angle of wall, ω the slope of the wall, ϕ the internal angle of friction, and β the angle of the backfill surface. Rankine provided a simplification of the above equations by assuming a frictionless and non-adhesive interface between the vertical wall and the bulk solid. Under such conditions the following stress ratios can be formulated (Rankine, 1856),

$$k_a = \tan^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right) \tag{14}$$

$$k_p = \tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right) \tag{15}$$

Note that k_a, shown in **Eqn. (14)**, is mathematically identical to **Eqn. (11)**. Rankine's formulation of k was one of the first theories of pressure for shallow bins, because the pressure distributions within these structures behave analogously to retaining walls. The assumptions fail when the bin height is increased, as the pressure exerted on the bottom and sides of the bin no longer fit the observations because the bin walls were assumed to be frictionless (Xu and Liang, 2022). This phenomenon is a property of bulk solids, but not considered until Janssen. Pleissner convincingly established the variation of k at the beginning of the 20th century. Many different numerical methods have adopted simplified versions of the Rankine formulation (14). For example, the Eurocode (European Standard EN 1991-4, 2006) used

$$k = 1.10\times(1-\sin\phi) \tag{16}$$

A mathematical model to account for vertical and radial changes in stresses and k is discussed in detail later through the work of Zhang et al. (1998a). They provided a literature review on the topic in the context of agricultural commodities, up to the publication date, and reported that there was no consistent agreement on how pressure, and the resulting k, varied radially. Rusinek investigated the variation in pressure of rape seeds using a hydraulic pressure transducer to determine the mean normal stress and mean shear stress on the wall, as well as the vertical pressure distribution, and found that k was significantly lower in the middle of the bin than the wall, and when wall friction increased, so did k (Rusinek, 2003). Horabik et al. discussed the behavior of k within bins (Horabik and Rusinek, 2002), and reported that the pressure ratio was mostly dependent on the internal angle of friction but also the shape of the seed (particles), and inversely related to moisture content; the experimental uniaxial compression test produced similar values to theoretical predictions given by **Eqn. (16)**. Qadir et al. (2010) showed experimentally that k increased with the ratio of the individual grain and diameter of the bin. Sun et al. presented four independent approaches to k and applied them to deep and shallow bins. The results were compared with experimental data and three national standards (Sun et al., 2018). They also compared different failure criteria to the typical Mohr-Coulomb criterion and recommended the Lade-Duncan (L-D) criterion to be used to calculate wall pressure. Uniaxial compression tests showed, for cereal grains, k decreased with increased moisture content (Horabik and Rusinek, 2002), whereas for calcareous sand, using a similar procedure, increased water content initially increased k, which reached a maximum and eventually declined again (Wang et al., 2020). Numerical studies that used discrete element methods suggested that k did not change with polydispersity (Wiącek and Molenda, 2014). Zhang et al. applied a two-parameter failure criterion that was typically used for soils to determine failure points (Zhang et al., 1994). When both parameters were nonzero, the lateral to vertical stress ratio increased with the vertical stress. At the same time, in other configurations, it was observed to be roughly equal to that determined from the Mohr-Coulomb failure criterion. Xu & Liang considered the elastic deformation of the bin walls in both static and dynamic states and demonstrated that the Rankine formulation of k would underestimate lateral forces (Xu and Liang, 2022). Back et al. studied the effect of friction on the pressure ratio k. Specifically, they focused on particle-particle friction, which the lateral pressure ratio was associated with through analytical arguments (Back, 2011). They established that k increased with particle–particle friction and packing.

2.3 After Janssen

The reaction to Janssen's work was immediate, and many people rushed to both validate and critique his analysis in the decades after the paper was published.

2.3.1 Wilfrid Airy: 1898

Airy noted at the beginning of his paper both the works of Isaac Roberts and Janssen, although he admitted he was not able to find any, "Tables of coefficients of friction of grain, wither on grain, or wood, iron, or brickwork..." (Airy, 1898). Moreover, the reference to Janssen was to the English translation of the abstract (Janssen, 1896), which was the reason why Airy did not have access to the friction coefficients. The translation only included the dimensions of the test cell, the equation describing the vertical pressure of grain, empirical data on corn depth and the resulting side pressure. Regardless, he noted these were reasonable investigations on the pressure of wheat (Janssen investigated corn) on the bottom of small bins. In the conclusion, Airy commented on the geometry of Roberts and Janssen's work and the limitation of "small bins", precisely how it would adjust the coefficient of friction on the rupture plane. Unfortunately, this topic was addressed by Janssen only in the original German publication. Moreover, Airy cited a textbook by Weisbach to describe "semi-fluids" for which, "the pressure on the side of a bin was the maximum pressure due to a wedge-shaped mass of the grain which might be supposed to separate from the general mass, and the angle of slope of the wedge-shaped mass which exerted the maximum pressure had to be determined" (Airy, 1898; Weisbach, 1849). The Mohr-Coulomb failure criterion was used as the basis for the pressure ratio. A slightly different approach was the specific application to "fluid–solids", which are known in the modern literature as "bulk solids". Airy was a contemporary of Janssen and took a different approach to modeling pressure, applicable to both shallow and deep bins. Airy defined the critical height h_L in terms of the coefficient of friction on the rupture plane $\mu' = \tan\phi$ as well as the coefficient of friction between the grain and the bin wall μ such that

$$h_\mathrm{L} = D\left[\mu' + \sqrt{\mu' \frac{1+(\mu')^2}{\mu+\mu'}}\right] \tag{17}$$

Airy then defined two regimes for shallow and deep bins.
Case 1: Shallow Bin, $h < h_\mathrm{L}$

$$\sigma_x = \rho y \left[\frac{1}{\sqrt{\mu'(\mu'+\mu)} + \sqrt{1+(\mu')^2}}\right]^2 \tag{18}$$

Case 2: Deep Bin, $h \ge h_\mathrm{L}$

$$\sigma_x = \frac{\rho D}{\mu'\mu}\left[1 - \frac{\sqrt{1+(\mu')^2}}{\sqrt{\frac{2h}{D}(\mu'+\mu)+1-\mu\mu'}}\right] \tag{19}$$

where h was the height of grain bulk solid pile, D the diameter of the bin and y the depth of grain measured from the surface of the pile. Airy started with the Mohr-Coulomb failure criterion, which was used to calculate the static condition of the wedge shape above the failure plane. The proof and resulting equations were complicated, and although the deep bin equation produces similar results to Janssen's. Airy's approach was not widely used due to the difficult calculation (Ketchum, 1911).

2.3.2 Milo Ketchum: 1907 and others

Milo Ketchum wrote a book "The Design of Bins, Walls and Grain Elevators", which exhaustively covered how to build all aspects of a modern elevator for the early part of the 20th century. It is an interesting text for a historical picture of the turn of the century in bin design and still provides a relevant pedagogical explanation of practices and equations. In chapter XVI, Ketchum discussed both

Janssen and Airy's proposed solutions to grain bin pressure (Ketchum, 1911). It is from this text that we find the formulation of **Eqn. (7)**. In Janssen's original text, the formulation was different, but only superficially. However, Ketchum's form is dominant because its derivation was clearly articulated in his text. Ketchum's work was important because it formalized Janssen's theory in a text discussing proper engineering practices and organized data that engaged with Janssen's theory. Ultimately, he concluded that Airy's equations were too complicated for practical use and recommended a graphical method for obtaining solutions. Ketchum provided summaries of many researchers who responded directly to Janssen's work, including Roberts, Prante, Toltz, Jamieson, Luft, Pleissner, Bovey, Lufft, and Pleissner. A.W. Roberts provided citations to all the original works discussed by Ketchum (Roberts, 1995; Roberts, 1998). This was the primary channel for understanding the development following Janssen's efforts.

As for the development after Janssen's efforts, a few are mentioned below to establish issues with Janssen's conclusions. A.W. Roberts provided a good summary of the work, which extended much beyond Ketchum's analysis (Roberts, 1995; 1998). Prante (1896) observed that the flow of bulk solids increased pressure as much as four times static values. Still, the results might have been unreliable due to the experimental apparatus, although the general trend was correct. These experiments established that grain removal should be symmetrical, otherwise this pressure increase on one side of the bin could bring about bin failure. Toltz (1903) utilized Prante's work while designing a grain elevator to account for fluid pressure within the facility. Toltz repeated Prante's experiment but changed the technique used to measure the deflection of steel plates and calculated the resulting pressure during filling and emptying. These results were considered reliable and confirmed that static and dynamic pressures were different, with dynamic pressures being significantly higher. Jamieson (1903) validated Janssen's model and calculations of lateral pressure ratio and also showed pressure changes in static and dynamic states with symmetric and asymmetric outlets. Furthermore, Jamieson's measurement device was complex, and allowed for both vertical and lateral pressures to be measured; his work added robustness to Janssen's theory. Pleissner (1906) conducted many experiments throughout 1902–1905 that showed k was not a constant in a bin and changed for different commodities. This was one of Janssen's assumptions, which most researchers adopted. Other experiments conducted by Bovey (1904) and Lufft (1902–1904) showed variability in k but less comprehensively than Pleissner. Finally, it should also be noted that outside of the enormous project of writing "The Design of Bins, Walls and Grain Elevators," Ketchum conducted experiments, which showed that the flow of bulk solids was independent of the grain height (the height of a bulk solid pile). This phenomenon is commonly known through Beverloo's model, published in 1961 but is also found in the second half of Hagen's work and is commonly called the Hagen-Beverloo Equation (Beverloo et al., 1961).

2.3.3 Andrew W. Jenike: 1961

Jenike was the first to mathematically describe the flow of bulk materials, which culminated in his book "Gravity Flow of Bulk Solids" (Jenike, 1961). While the topic of this paper is static pressure, Jenike developed essential descriptions of flow that determined what future state a system would take and is thus relevant to this review. Jenike's work drew heavily on his contemporaries as well as investigations into mathematical descriptions of plasticity. In addition to new analytical theories and methods, the advent of computers would make it possible to carry out otherwise "insoluble" mathematical problems. This comment hinted at the future of bulk solid research and where one finds the forefront of research today, introduced through the work of P.A. Cundall and O.D.L. Strack, who developed the discrete element method (Cundall and Strack, 1979). The work "Gravity Flow of Bulk Solids" was outlined into six parts: 1) the nature of the yield function; 2) differential equations that described steady-state flow and produce pressure, density, and velocity fields; 3) the initiation of flow, which was called incipient failure; 4) necessary criteria for the bulk solid to flow; 5) apparatuses to test flow properties of bulk solids; and 6) experiments and applications of this flow theory. Jenike categorized different types of bulk solid flows and how the bin/hopper geometry and other properties could influence flow types. Jenike noted that there existed no perfect bin; each bin had its own unique advantages and disadvantages. In part VI, Jenike mentioned both Janssen and Ketchum's work as being the most important contributions to the knowledge of bulk solids at the turn of the century. Jenike also produced several other important publications. One was written with J.R. Johanson and J.W. Carson that described bin design, mass flow, and funnel flow in a four-part series (Jenike et al., 1973a; 1973b; 1973c). Moreover, Jenike summarized bin loads under different flow states: static, during flow, and steady flow (Jenike and Johanson, 1969). These later publications were often cited in both research and engineering practices because they included design formulas and examples of calculations.

2.3.4 Walker: 1966

Walker investigated pressure and arching in hoppers, using Janssen's model and calculating a k value in both static and flow states. His work also was an attempt to determine whether a specific hopper would experience blockage through arching of powder across the outlet. In addition, he investigated the role cohesion played in the gravity flow in a hopper (Walker, 1966).

2.3.5 Marcel Reimbert and André Reimbert: 1976

Marcel and André Reimbert wrote a book titled "Silos-Theory and Practice", which presented a new model for bin pressures (Reimbert M. and Reimbert A., 1976). The book was written with the same spirit as Milo Ketchum's "The Design of Bins, Walls and Grain Elevators". It was ultimately interested in the construction of grain elevators but diverged by presenting a new theory of bulk solid phenomenon to inform calculations instead of aggregating established theories. The book also offered critiques on the traditional Koenen-Janssen approach and noted that the Janssen method needed to assume a constant k. Additionally, Ketchum suggested k as a function of bin geometry and material depth. The model proposed in the book produced similar results to Janssen's. However, a significant difference between the two was that Reimbert et al. used an empirical equation (hyperbola) to describe the change in pressure as a function of depth. The equation was developed for cylindrical silos and can also be used for n-sided polygons. Furthermore, while the authors argued that k changed throughout a bin, they still assumed it to be constant in calculation, and the value they took ("the characteristic abscissae") was calculated at the maximum vertical pressure, which resulted in

$$A = \frac{D}{4\tan\phi'\tan^2\left(\frac{\pi}{4}-\frac{\phi'}{2}\right)} - \frac{h}{3} \tag{20}$$

where D the diameter of the bin. The maximum stress was given by

$$\sigma_{\max} = \frac{\rho g D}{4\tan\phi'} \tag{21}$$

with the horizontal stress,

$$\sigma_x = \sigma_{\max}\left[1-\left(\frac{y}{A}+1\right)^{-2}\right] \tag{22}$$

where y was the depth of the material.

3. Comparison

Many models have been proposed in the past 100 years and the immediate differences between these models are not intuitive. A quantitative comparison was conducted to compare the results between Hagen, Janssen, Airy, and Reimbert. Each model uses the same properties as shown in **Table 1**.

Hagen's work attempted to recreate the "Janssen effect" using a parabolic increase in pressure until a maximum was reached, after which it remained constant. This piece-wise approach used a quadratic equation to describe the increase in pressure, and constant once the maximum was reached, represented by the dashed line in **Fig. 1**. Hagen's model is the crudest of the historical methods but provided an essential first step; subsequent models predicted very similar behavior giving a more gradual increase than Hagen. Janssen's model predicted an exponential change in pressure with the material depth that approached an asymptote. Airy's model used a complicated geometric approach and applied the knowledge developed by Coulomb and others in earth pressure theory to bulk solids in bins. Both Airy and Hagen's models are separated by a transition point in which the pressure changes discontinuously. Reimbert's equation found the maximum pressure at the bottom of the bin and modeled the change in pressure using a hyperbolic approach. Reimbert attempted to create a new variable for k that could account for more factors than the other methods. For example, Janssen assumes a lateral pressure ratio to remain constant within a particular bin, but we would expect k to vary from bin to bin as a function of geometry.

Our analysis of these models at different depth/diameter ratios revealed that the differences in pressure predictions between the models can be divided into three main regions: low (0–1.5), intermediate (1.5–3), and high (3–5). At the low h/d ratio region, the equations exhibited the most variation between each other. The following percentage differences were observed: Hagen vs Reimbert, 16.0 %; Hagen vs Janssen, 31.3 %; Hagen vs Airy, 64.6 %; Reimbert vs

Table 1 Model variables and values used to generate **Fig. 1**, with the bin diameter d, grain height h or y, mass density ρ, stress ratio k, coefficient of friction μ and the internal angle of friction ϕ.

Parameter	Value	Unit
d	6	m
h or y	30	m
ρ	770	kg/m^3
k	0.4	N/A
μ	0.7	N/A
ϕ	30	deg

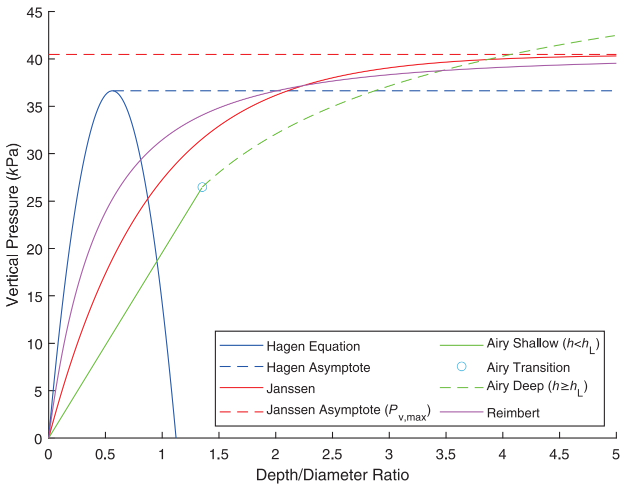

Fig. 1 Comparing historical bin pressure models. (The data are available publicly at https://doi.org/10.50931/data.kona.23741598)

Janssen, 15.6 %; Reimbert vs Airy, 50.0 %; and Janssen vs Airy, 35.1 %. In the intermediate region, Reimbert and Janssen closely approached the Hagen asymptote, with Airy's equation being 10.0 % lower during this intersection. At the extreme end of the high *h*/*d* ratio region, Reimbert and Janssen were very close, within 2.0 % of each other, while Airy and Hagen's asymptote were 5.1 % above and 7.6 % below them, respectively. It is important to note that there is a 9.9 % difference between the Hagen and Janssen asymptote, with Janssen predicting greater pressure.

4. New developments in static pressure theories

While the Janssen equation produces satisfactory results for static conditions if the "right" parameters are selected, it does not represent the full pressure field present within a bin. Many new models and approaches to the pressure field within a bin have provided new insights. Such modern approaches include elastic theory (Bräuer, 2006; Ovarlez and Clément, 2005; Schillinger and Malla, 2008), ordinary stress linearity (OSL) and incipient failure everywhere (IFE) models (Vanel et al., 2000), microscopic theory (Xu et al., 1996), the principal stress cap approach (Matchett, 2006), as well as a plethora of numerical methods (Chen, et al., 1999).

4.1 Investigations and modifications of Janssen's model

Rigorous validation of Janssen's model occurred within the first decade of its publication (Ketchum, 1911; Roberts, 1995). Many modern studies have attempted to validate or modify Janssen's model. Di Felice et al. (Di Felice and Scapinello, 2010) highlighted several works in the past decade and identified a literature gap that questioned important parameters in the Janssen model: μ and k. Their work showed that while many studies determined the numerical values of these parameters, their physical interpretation was not as clear. Moreover, empirically their study showed some discrepancies when measuring these parameters. While De Felice et al. highlighted the need to investigate Janssen's assumptions, others have taken up this task. Janssen assumed bins to be a static and closed environment, where no forces are transferred from outside the system besides the self weight of stored materials. This is not the case for most bins, which experience external forces from their environments, such as vibrations caused by machinery (Körzendörfer, 2022), earthquakes (Silvestri et al., 2012), intentional flow-inducing inertia forces (Pascot et al., 2020) and bin honking (Buick et al., 2004). Bertho et al. looked at the applicability of Janssen's model in dynamic environments, provided an overview of the current research, and considered the effect of wall movement on packing (Bertho et al., 2003). They found the classical Janssen model was valid for bin walls that experienced several centimeters per second movement. Moreover, Windows-Yule et al. investigated horizontal wall movement and found that the Janssen model was also valid for a wide range of dynamic movement (Windows-Yule et al., 2019). It is worth noting that this topic was often investigated by numerical methods, an important tool for understanding bulk solids but a topic outside the scope of this paper.

Janssen's model is a macroscopic model, which treats the material as a continuum instead of a collection of particles. Bratberg et al. examined the threshold where microscopic behaviors turn into macroscopic behaviors through narrow granular columns (Bratberg et al., 2005). They found that Janssen's model could not account for small container diameters. Similarly, for shallow granular pile heights, an interesting result was discovered by Mahajan et al. (Mahajan et al., 2020)—a "reverse Janssen effect". Specifically, the wall frictional forces could become compressive, effectively reversing the usual Janssen effect. Under this condition, the walls increase the effective mass of the bulk material at the bottom, which requires modification to Janssen's equation to predict the changes in effective mass as a function of height. This reverse effect was only observable when the height of a bulk solid pile was relatively small, i.e., $h \le 30d_G$ with d_G the diameter of a single particle. Thus, we do not generally expect to see this effect on large scales such as within bins. The continuum approach is typically used in analytical models, whereas numerical models can be applied to both continuum and particulate materials (Chen et al., 1999; Schulze, 2008).

Many works focused on addressing the inadequacies of Janssen's assumptions. Some examples include variable bulk density due to particle packing (Vanel and Clément, 1999; Haque, 2013; Landry et al., 2004), variable angle of repose (Pont et al., 2003), horizontal bins (Tang et al., 2021), pressures in the hopper section (Walker, 1966), obstacles in bins (Endo et al., 2017), elastic deformation of bin walls (Xu and Liang, 2022), moisture content (Zhang et al., 1998b; Chen, et al., 2020), temperature (Lapko et al., 2003), shallow bins (Ooi and Rotter, 1990; Xu and Liang, 2022), as well as loading and unloading stresses (Walters, 1973), among others (Ayuga, 2008). These modifications are relevant because they point toward higher fidelity models that can better describe physical systems. Such modifications are potentially important for future industrial applications as well as advancing development of state-of-the-art models. While it is not realistic to review the studies that address all the factors listed above, this paper has selected one significant topic to discuss, namely variable stress fields.

4.2 Variable stress fields

The most fundamental and controversial assumption in

Janssen's theory is the constant lateral to vertical pressure ratio, k (discussed at length in **Section 2.2**). This ratio is fundamentally dictated by the stress field in the bulk solid mass. Many experiments have shown that this ratio is a function of material properties and location in a bin (Chen, et al., 2020; Liu et al., 2021). In this section, the stress field is investigated analytically in a 2D Cartesian coordinate system (x,y), where the y direction points "downward," with $y = 0$ representing the top of the bulk solid surface. Deriving Janssen's equation in the Cartesian case requires the use of a "slip-state" condition throughout the bin such that $\tau_{xy} = \mu\sigma_x(y)$ at the wall. In terms of vertical stress, we have

$$\tau_{xy}(y) = \mu\sigma_x(y) = \mu k \sigma_y(y) \tag{23}$$

The equilibrium in the vertical direction can be expressed as:

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = g\rho \tag{24}$$

If the stresses are assumed not to vary in the horizontal direction, as in Janssen's theory, or $\tau_{xy} = \tau_{xy}(y)$, the partial derivative of shear stress with respect to x vanishes,

$$\frac{\partial \tau_{xy}}{\partial x} = 0 \tag{25}$$

This means the equilibrium equation in the vertical direction is simplified as

$$\frac{\partial \sigma_y}{\partial y} = g\rho \tag{26}$$

which has the solution of

$$\sigma_y = g\rho y \tag{27}$$

This indicates that the bulk solids behave like a fluid if the shear stress varies only in the vertical direction. Here, the stress changes with depth, just as a fluid does. In this case the wall friction has no effect on the vertical stress in the material. Mathematically, this vanishing of the partial derivative of the shear stress can be observed due to the relationships presented in **Eqn. (25)**. Let's now consider a variable stress field in the horizontal direction. Specifically, we postulate a shear stress which varies linearly in the horizontal direction,

$$\tau_{xy}(x,y) = \frac{x}{b}\mu\sigma_x(y) \tag{28}$$

where b is the radius of the bin. Now the partial derivative term is expressed as

$$\frac{\partial \tau_{xy}}{\partial x} = \frac{\mu}{b}\sigma_x(y) \tag{29}$$

and the equilibrium equation has the form of

$$\frac{\mathrm{d}\sigma_y}{\mathrm{d}y} + \frac{\mu k}{b}\sigma_y = g\rho \tag{30}$$

This equation form is similar to that of Janssen's differential equation. This means that in order to account for the Janssen effect, the shear stress in $2D$ must include radial dependence. It should be noted that exchanging b to R_H in **Eqn. (28)** would identically result in the Janssen equation. The difference between the granular and liquid-like behaviors occurs because the effect of wall friction is not included in the problem when the derivative term drops out. Deriving the Janssen equation requires a specific choice of shear stress that links the friction coefficient and the vertical stress through the horizontal to vertical stress ratio $\sigma_x(y) = k\sigma_y(y)$. Geometrically, this term naturally arises in cylindrical coordinates (r, θ, z) due to the radially dependent volume of infinitesimal cylindrical elements $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$. However, this geometrical effect does not occur in Cartesian coordinates (x, y, z), in which the volume of all infinitesimal elements are independent of the radial coordinate.

Radial variation of the shear stress can be traced back to 1948 (Jaky, 1948). In relation to flow problems this term shows up in the work of McInnes (1968) and Savage (1967). Later, it was used by Zhang et al. (1998a) and Milleteet al. (2006) to develop a theory of radial stress behavior in a $2D$ Cartesian coordinate system and later to expand to $3D$ cylindrical coordinates (Rahmoun et al., 2008).

We compare the results of Zhang et al. (1998a) with the usual Janssen approach for a hypothetical bin of 5 m in diameter (**Fig. 2**). Zhang et al.'s model accounted for the non-uniformity of stresses in the horizontal and vertical directions, whereas Janssen's work only considered change in the vertical direction. Additionally, in the Zhang model k varies throughout the bin, whereas variation in k is neglected by Janssen. Zhang's model suggests that higher stresses occur at the center of a bin. This may have design implications for airflow in aeration and drying.

To quantify the differences between the Zhang and Janssen models, we calculated the percentage differences between their stress values at various depths and distances from the bin center. The analysis revealed a range of percentage differences between the two models in both horizontal and vertical stress distributions. The vertical stress distribution exhibited more variation than the horizontal, so we focused on this set of data. At the most distant point from the center of the bin, Zhang's model predicted less pressure than Janssen, ranging from 40.0 % to 24.0 %, with the maximum discrepancy occurring at the 1 m depth and decreasing afterward without a clear trend. In contrast, Zhang's model predicted higher pressure at the center of the bin, with differences ranging from 22.5 % to 7.6 %; the maximum discrepancy occurred at the 1 m depth and decreased with depth. At all depths, Zhang's pressure crossed

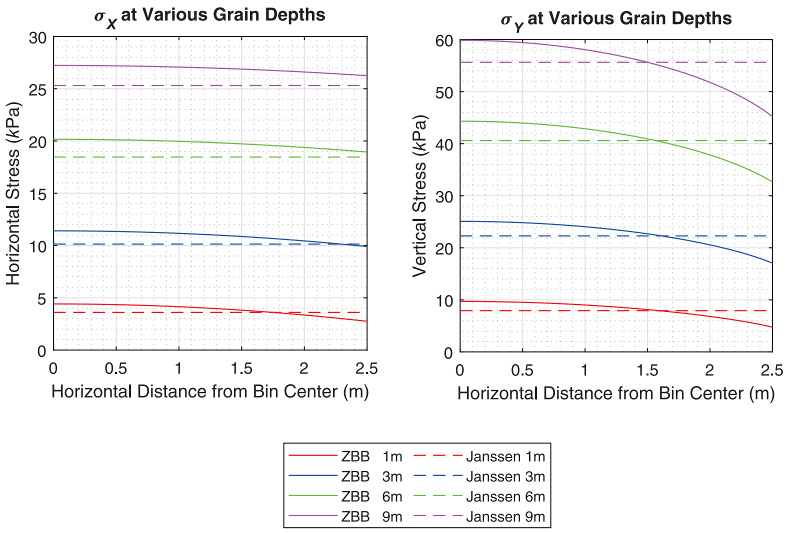

Fig. 2 Differences between the linear stress variation used by Zhang, Bu and Britton (Zhang et al., 1998a) and the planar Janssen result using the Rankine coefficient (**Eqs. 30** and **11**) assuming a bin radius of $b = 2.5$ m, uncompacted mass density $\rho = 834$ kg m^{-3}, coefficient of static friction $\mu = 0.36$ and internal angle of friction $\phi = 22°$ using $h = 1$ m. Bin depth is given in the legend. Change in the radial dependence is introduced by the linear modification to Janssen's equation. (The data are available publicly at https://doi.org/10.50931/data.kona.23741598)

over Janssen's between 1.5 and 1.6 meters from the bin center, and generally, as depth increased, this crossover occurred closer to the center of the bin. This indicated that the Zhang model's predictions could significantly deviate from those of the Janssen model, depending on the depth and distance from the bin center. A consistent trend observed across various depths and distances was that Zhang's model predicted lower pressure and greater discrepancy on the outside, while the center experienced higher pressure and relatively less discrepancy compared to Janssen's model. This finding further emphasized the importance of considering non-uniform stress distributions in grain bin design for optimal aeration and drying performance.

The work of Rahmoun, Millet and de Saxcé (Millet et al., 2006; Rahmoun et al., 2008) included significant expansions over the original Janssen model, specifically applicable to cohesive powder materials. The analysis in Millet et al. (2006) for a planar bin with the $2D$ Cartesian equilibrium equations (**Eqn. (8)**) resulted in a differential equation for the angle between the horizontal axis and the principal stress direction. The differential equation was solved analytically. The model was numerically compared to Janssen's equation for low cohesion ~ 10^{-6} Pa and the results were found to agree for the passive and active states. A similar approach was taken in Rahmoun et al. (2008) in which the cylindrical equilibrium equations (**Eqn. (9)**) were used to produce a more general $3D$ result. The resulted model could be solved analytically for cohesionless materials, but a numerical approach was required when finite cohesion was included.

An important model used to complement the Mohr-Coulomb criterion is known as the incipient failure everywhere (IFE), or the rigid-plastic model (Nedderman, 1992; Wittmer et al., 1996, 1997). The IFE model assumes that the material is on the edge of failure throughout the entire bin. This means that through each point there is some plane over which the shear force is linearly related to the normal force. The variable vertical to horizontal stress ratio given by the IFE model has the form of Wittmer et al. (1996)

$$\alpha(\beta) = \frac{(1+\sin^2\phi) \pm 2\sin\phi\sqrt{1-\left(\dfrac{\cot\phi\,\tau_{xy}}{\sigma_x}\right)^2}}{\cos^2\phi} \tag{31}$$

where α is the vertical to horizontal stress ratio (k^{-1}) and β is the shear stress to horizontal stress ratio,

$$\sigma_y = \alpha\sigma_x \tag{32}$$

$$\tau_{xy} = \beta\sigma_x \tag{33}$$

Substituting **Eqns. (32)** and **(33)** into **Eqn. (31)** yields

$$\alpha(\beta) = \frac{(1+\sin^2\phi) \pm 2\sin\phi\sqrt{1-\dfrac{\beta^2}{\tan^2\phi}}}{\cos^2\phi} \tag{34}$$

The $\alpha(\beta)$ has real solutions if $\beta \le \tan\phi$. This condition is upheld because the stress ratio β is a maximum at the bin wall $\beta|_{r=b} = \mu$, so we have generally $\beta \le \mu \le \tan\phi$. Moreover, under static conditions the vertical stress should exceed the horizontal stress, so we expect $\alpha > 1$, which selects the + sign in the quadratic solution. In this case, the variation in α accounts for the change in stress across the bin radius. Similar to the approach outlined above, Millet et al. (2006) considered the variation in radial stresses for cohesive materials.

After Jenike outlined proper designs to avoid arching and funnel flow, hoppers became a common addition to bins. The stress field in a hopper is variable and Janssen's equation is not applicable to hoppers. Walker (1966) and Takami (1975) both used force-balance conditions on a conical slice to describe the stress distribution within a conical hopper and showed vertical and horizontal variations in stresses using Rankine's formulation of *k*. This work, similar to Janssen's theory, considered forces on a planar differential slice, which was assumed axisymmetric, which is a reasonable assumption for most cases, but not all. Some bins have eccentric hoppers, which produce a non-axisymmetric distribution of material and stresses. While there have been many studies into eccentric discharge of bins in terms of improper flow, there have been far fewer on structural loads (stresses), but this has changed in the past two decades (Wojcik et al., 2003; Vidal et al., 2006, 2008). A detailed study was reported by Ramírez et al. (2010), which provided a thorough discussion of the benefits of eccentric hoppers, experimental work, and several numerical models. This work inspired Matchett's analytical model based on the "principal stress cap" approach, which was capable of accounting for eccentricities in bin loads, as well as symmetrical loads (Matchett, 2006; Matchett, 2020; Matchett and Close, 2021). This research provided a fundamental start to designing safe eccentric hoppers.

4.3 Variable density

Janssen's equation assumed the density of a bulk solid to be constant throughout the entirety of a bin. There have been substantial attempts at numerically incorporating packing into pressure models and considering the effect of particle impact forces on spatial arrangements of particles (Volfson et al., 2003; Landry et al., 2004; Umbanhowar and Goldman, 2010; Petingco et al., 2022). There are some direct modifications to Janssen's model to account for density variations, with varying complexity. Vanel reported that packing could affect *k* and these effects could be quite significant when the porosity changed from 0.43 to 0.39 (Vanel and Clément, 1999). They proposed a modification to Janssen's model by introducing a fit parameter to account for inhomogeneous density. Cheng et al. directly modified Janssen's equation to account for changes in density and investigated bins with hopper sections (Cheng et al., 2017). They related the degree of packing to the principal stress through a quadratic equation and addressed gaps and ambiguous understandings of pressures in the literature. They concluded that bulk density increased rapidly with the material depth in the upper cylindrical part of the bin but decreased slowly in the hopper. Haque utilized hoop stress and hoop pressure, which represent the circumferential stress and force exerted by stored material in a cylindrical structure, to analytically describe packing within a bin (Haque, 2013). Developing a modified Janssen's equation, Haque's model accounted for variations in bulk density when calculating vertical and horizontal pressures within a bin, offering a more comprehensive understanding of the forces involved in grain storage.

4.4 Non-continuum approaches

4.4.1 Microscopic theory

Bulk solids are discontinuous in nature and microscopic theory may provide a better alternative to the continuum approach such as Janssen's, but with greater complexity. Bratberg et al. investigated the failure of Janssen's theory as the bin and material scale transitioned from macroscopic to microscopic. A few researchers have investigated the bin pressure from the microscopic angle. Xu et al. developed a bin pressure model accounting for the discontinuous nature of bulk solids (Xu et al., 1996). Their model was based on a more general microscopic approach by Granik & Ferrari that used principles of thermodynamics to investigate a system of doublets (pairs of adjacent particles; Granik and Ferrari, 1993). The particles modeled by Xu et al. were idealized spheres in hexagonal packing and held within 2*D* frictionless walls. Under a pair of contact forces within a doublet, the two particles experienced microstrains due to deformation, rotation, and slipping. Microstress (elongation, compression, torsion, and shearing) were calculated from the contact forces, but torsion and shear microstresses were found to be negligible. Their model results showed that Janssen's results were a special case of their microscopic theory. Moreover, *k* was only a constant value when the model used frictionless walls and rigid particles. Xu had a deeper examination of the application of microscopic theories in his Ph.D. dissertation (Xu, 1966). Xu et al. demonstrated how this microscopic theory could be used to predict moisture induced stresses within the mass of hygroscopic bulk solids. Zhang et al. further developed this theory to explain discharge loads in bins (Zhang and Britton, 2003). Ferrari et al. compiled a review of doublet microscopic theory (Ferrari et al., 1997). The strength of this theory was the ability to analytically describe the discontinuous nature of particle assemblies, something that was beyond the scope of the continuum approach.

4.4.2 Granular elasticity

Jiang and Liu gave a brief overview of granular elastic theory (Jiang and Liu, 2007a), which covered the original work done by Jiang and Liu (2003). Granular elasticity was distinct due to its definition of strain, which has two parts. The first accounted for particle deformation (elastic). The second accounted for irreversible rolling and slippage (a process that also causes heat) of particles and was considered plastic. The irreversible plastic phenomenon was the main contributor to the overall deformations. The energy within the material under loading could be articulated as a function of strains and static stresses. Elastic strains were used because particles are compressed and sheared, which stores energy reversibly. These models are based on a relationship between elastic energy, static stress and the deformation of particles. The stress distributions in a granular material could be determined the same way as in elastic media. Granular elasticity could account for a bulk solid's yield, volume dilatancy, and shear-induced anisotropy. Jiang & Liu discussed the applicability of this theory to bins and compared it to Janssen's method. They found that bins were a good fit for granular elastic theory because all six stress tensor elements could be calculated. Bräuer et al. validated granular elasticity for its application to bin stress (Bräuer, 2006). Their model showed a saturation pressure that matched Janssen's. The remaining components of the stress tensor were also computed, which Janssen's theory could not perform. They assumed a constant k and found little variation in vertical stress radially. Other interesting works on granular elasticity included a slow movement of deforming particles (Jiang and Liu, 2007b), which accounted for the infamous stress dip in granular piles (Krimer et al., 2006), showing good agreement with experimental data (Jiang and Liu, 2008). Sun et al. applied granular elasticity to the mesoscopic scale using numerical methods, which examined packing, pressure, and force network structures (Sun et al., 2015). While the study was focused on mesoscopic objects, there was a possibility that this approach could be applied to many discontinuous systems.

5. Concluding remarks

New perspectives and viewpoints are often required to solve stubborn, longstanding problems. This was certainly the state of the field when Janssen's approach produced a revolution in describing and predicting the behavior of granular materials in bins. Even in the modern literature, Janssen's model forms the cornerstone of our understanding of pressures in bins. Modifications to Janssen's model have been sought to generalize it for describing the observed phenomena to higher accuracy, but the starting point is usually Janssen. Otherwise, it is the first validation case to test a new theory against. This ubiquity is a monument to Janssen's success. Just as a new perspective spurred the revolution that led to Janssen's insights, a survey of the literature reveals many new promising approaches.

Despite these successes, there are several limitations of Janssen's model. First, the vertical stress is assumed to be constant across a bin. This is not generally true as the pressure is expected to vary from the center to wall of a bin. Second, the horizontal stress is assumed to be proportional to the vertical stress through the factor k, which is constant throughout a bin. Finally, the density is considered constant throughout a bin. The first of these assumptions is incorrect. The second statement is ad hoc, and generally it should require some deeper theoretical justification. The third assumption is generally incorrect.

In recent years the computational power available to perform DEM simulations has significantly increased, and numerical methods used to simulate granular properties have been extensively studied. The discussion of material properties acquisition and standardization has made DEM a more viable approach for solving bulk solid problems (Xu et al., 2002; González-Montellano et al., 2011; González-Montellano et al., 2012; Horabik and Parafiniuk, 2016). Generally, DEM simulations need to limit the number of particles for computational purposes, and the particle shape sometimes needs to be crudely approximated. However, strategies have been developed to mitigate these limitations (Ramírez-Gómez, 2020). DEM provides the numerical revolution analogous to the analytical revolution provided by Janssen.

Data Availability Statement

The data from the pressure models are available publicly in J-STAGE Data (https://doi.org/10.50931/data.kona.23741598).

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Jitendra Paliwal

Dr. Jitendra Paliwal is a Professor of Biosystems Engineering whose expertise lies in the storage, handling, and quality monitoring of cereal grains, oilseeds, leguminous crops, and their derivatives. His research on developing hardware and software solutions is widely referred to by the designers of grain quality monitoring and assessment instruments. In 2018, he was part of a distinguished team of industry and academic researchers, who won the American Society of Agricultural and Biological Engineering's AE50 award for adapting a cancer detection imaging technique for remote monitoring grain bins.