A Spheroid Model for the Role of Shape in Particle Size Analysis†

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Abstract
Standard procedure in particle size analysis is to express size as an equivalent sphere diameter. As a consequence, results obtained by procedures based on different kinds of response give different results. An alternative approach is presented based on a spheroidal shape which, depending on elongation can represent shapes ranging from rods, through spheres to discs. Particle size and shape are then presented as volume diameter and elongation. Applications to commonly used procedures are presented. For particles of similar shape, i.e., elongation independent of size, the results of procedures such as sieving, microscopy and sedimentation show a simple shift in apparent size with the form of the distribution unaffected. In the case of methods based on radiation scattering, orientation effects lead to an apparent size distribution even for identical particles. Experimental results on glass spheres, crushed quartz particles and fine kaolin are shown to be consistent with the spheroid model. Analysis of the quartz data by a variety of techniques gave consistent values for the volume diameter and elongation. Light scattering measurements on a series of narrow sieve fractions of the quartz particles showed no variation in elongation with size.

Keywords: particle size and shape, spheroidal shape, volume diameter, elongation, sieving, image analysis, sedimentation, light scattering, optical and electrical counters

1. Introduction
It is well known that particle size measurements are affected by particle shape. The results of such measurements are generally expressed in terms of an “equivalent sphere diameter” defined as the diameter of a sphere that would have the same response to the measurement procedure as the actual particle. In other words the procedure involves the assumption that the particles are spherical in shape. In practice, of course, real particles are rarely truly spherical and often depart significantly from that idealized shape. Departures can include relatively small-scale irregularities and macroscopic differences in axial dimensions—length, breadth and thickness. The effects of surface irregularities presumably have some effect on size measurements, but such effects are likely to be small and very difficult to quantify. Axial ratios, on the other hand, for shapes varying from needles to platelets can be expected to have major effects.

An important consequence of the effects of shape is that different measurement procedures typically yield different results, sometimes significantly. For routine applications such as material characterization and product specification, it is commonly sufficient to avoid the problem by the consistent use of the same analytical procedure. However, if shape is important to an application or if two or more procedures are required for complete analysis of a broad size distribution, the shape effect should be taken into account.

While the discrepancies between size distributions obtained using different methods can lead to problems in specification and the comparison of results, these differences also contain information, albeit indirect, on shape itself. The differences in apparent mean size between two methods can be used as a measure of departure from the idealized spherical shape. Comparison of the complete distributions from several methods may provide further information on the kinds of shapes involved.

The analysis presented in this paper is based on the use of the spheroidal shape as an alternative to the simple sphere to evaluate the role of shape in some widely-used procedures for particle size analysis and to investigate the potential for using differences in apparent size distributions to extract shape information.
2. Spheroid Geometry

A spheroid is simply a sphere that has been stretched or compressed along one axis and is represented by the rotation of an ellipse about that axis. The particle profile is a circle of radius $R$ along that axis and an ellipse of radius $B$ perpendicular to the axis as illustrated in Fig. 1. The elliptical profile is described by:

$$\frac{x^2}{R^2} + \frac{y^2}{B^2} = 1$$

Defining the elongation $E$ as the ratio $B/R$ the shape is that of an oblate (flattened) spheroid for $E < 1$, a prolate (elongated) spheroid for $E > 1$ and a sphere for $E = 1$.

Particle volume is a unique measure of size, independent of shape or orientation and is commonly represented by the volume diameter $d_v$, defined as the diameter of a sphere with the same volume $V$ as the particle. For a spheroidal particle

$$V = \frac{4}{3} \pi R^2 B = \frac{4}{3} \pi R^3 E$$

$$d_v = \left(\frac{6V}{\pi}\right)^{1/3} = 2RE^{1/3}$$

Surface area $A_s$ can be represented by the surface area diameter $d_s$, defined as the surface area of a sphere with the same surface area as the particle. Referring to Fig. 1, the area $ds$ of a surface element at $x$ is given by

$$ds = 2\pi y \frac{dy}{\sin \theta} = 2\pi \left(1 + \frac{1}{\tan^2 \theta}\right) y dy$$

By differentiation of Eqn. 1

$$\tan \theta = -\frac{dy}{dx} = \frac{x}{E^2 y}$$

from which

$$y dy = -\frac{1}{E^2} x dx$$

Also, from Eqn. 1

$$E^2 y = B^2 - x^2$$

Substitution from Eqns. 5, 6 and 7 into Eqn. 4 leads to

$$ds = -\frac{2\pi}{E^2} \left[\sqrt{E^2 B^2 + x^2 (1 - E^2)}\right] dx$$

Integration of Eqn. 8 leads to the following expression for the total surface area

$$S = 2\pi R^2 f(E)$$

where

$$f(E) = f_0 = 1 + \frac{E^2}{\sqrt{1 - E^2}} \ln \left[1 + \sqrt{1 - E^2}\right]$$

for oblate, $E < 1$ and

$$f(E) = f_p = 1 + \frac{E^2}{\sqrt{E^2 - 1}} \arcsin \left[\sqrt{E^2 - 1}\right]$$

for prolate, $E > 1$.

The surface diameter is defined by

$$d_s = \sqrt{\frac{S}{\pi}}$$

so that

$$d_s = R\sqrt{2f(E)}$$

3. Application to Particle Size Analysis

All size analysis procedures are affected in some way by particle shape. Effects on some widely used procedures are evaluated below.

3.1 Sieving

Sieve size is generally defined as the limiting size at which a particle can pass through a square aperture with side dimension $d_A$. A prolate spheroid can pass through a sieve opening provided its smaller dimension $R$ is less than half the opening size, i.e.,

$$d_A = 2R$$

In this case, the sieve size is, in principle, independent of the larger dimension $B$. In practice, of course, the probability of achieving the appropriate orientation by a random shaking operation becomes increasingly small for highly elongated particles.

Oblate spheroids ($E < 1$) whose larger dimension, $2R$, is somewhat larger than the sieve aperture can pass if properly oriented. The minimum sieve opening size is determined by

$$d_s = R\sqrt{2f(E)}$$
as shown in Fig. 2.

It can be seen from the geometry of the figure that \( X = Y = x + y \) where \( x \) and \( y \) are the coordinates of the point of tangency of the elliptical particle profile and the sieve opening. Thus,

\[
d_A = \sqrt{2} (x + y)
\]  
(15)

The line \( XY \) is tangent to the surface of the particle and, for a square opening, has a slope \( dy/dx \) of minus one. Differentiating Eqn. 1,

\[
\frac{dy}{dx} = -\frac{B^2 x}{R^2 y} = -E^2 \frac{x}{y} = -1
\]  
(16)

i.e.,

\[
y = E^2 x
\]  
(17)

Combining Eqsns. 15, 16 and 17,

\[
d_A = R\sqrt{2(1 + E^2)}
\]  
(18)

It is convenient to express the sieve size \( d_A \) relative to the volume diameter \( d_v \) which is given by Eqn. 3. Combination of Eqsns. 3 and 14 leads to

\[
d_A = \frac{d_v}{E^{1/3}}
\]  
(19)

for a prolate spheroid \( (E \geq 1) \). Similarly, Eqsns. 3 and 18 give

\[
d_A = \frac{d_v}{E^{1/3}} \sqrt{\frac{1 + E^2}{2}}
\]  
(20)

for an oblate spheroid \( (E \leq 1) \).

Eqn. 20 can also be applied to an ellipsoidal particle using the ratio of the intermediate to the smallest of the three principal axes in place of \( E \). As in the case of prolate spheroids sieve size for ellipsoidal particles is, in principal, independent of the longest axis but in practical applications will generally be limited for highly elongated particles.

An example of the effect of elongation the sieve/volume diameter ratio is presented in Fig. 3. It can be seen that relative to particle volume, sieving gives an overestimate of size for ellipsoids or oblate spheroids and an underestimate for prolate spheroids.

3.2 Microscopy/Image Analysis

Direct measurements of particle dimensions by microscopy are generally performed on particles in a stable configuration on a substrate. Image analysis typically provides a size estimate based on the projected area \( A_{ps} \) of the image. The corresponding size \( d_{ps} \) is defined as the diameter of a circle with the same projected area as the particle. Thus,

\[
d_{ps} = 2\sqrt{\frac{A_{ps}}{\pi}}
\]  
(21)

For prolate spheroids, the projected area is

\[
A_{ps} = \pi RB = \pi R^2 E
\]  
(22)

so that

\[
d_{ps} = 2R\sqrt{E} \quad (E \geq 1)
\]  
(23)

For oblate spheroids,

\[
d_{ps} = 2R
\]  
(24)

Expressed relative to the volume diameter using Eqn. 3

\[
d_{ps} = d_v E^{1/6} \quad (E \geq 1)
\]  
(25)

and

\[
d_{ps} = \frac{d_v}{E^{1/3}} \quad (E \leq 1)
\]  
(26)

![Fig. 2 Limiting sieve size for an oblate spheroid.](image)

![Fig. 3 Effect of elongation on sieve size relative to volume diameter.](image)
It follows that particle size relative to volume diameter for spheroids is always overestimated by microscopy for both oblate and prolate types. As expected, the effect is most pronounced for flattened (oblate) particles.

3.3 Sedimentation

Particle size estimates by sedimentation are generally presented in terms of the Stokes diameter $d_s$ defined as the diameter of a sphere that has the same (terminal) free settling velocity in a fluid (under laminar flow conditions) as the particle. At the terminal velocity, the buoyant weight of the particle (proportional to particle volume, i.e., $d_v^3$) is balanced by the hydrodynamic drag force acting (proportional to the so-called drag diameter, $d_d$). The Stokes diameter is then defined as

$$d_s = \sqrt{\frac{d_v^3}{d_d}}$$ (27)

Since, especially under laminar-flow conditions, the fluid drag results primarily from shear at the particle surface, it is often assumed (Allen, 1997) that the drag diameter $d_d$ is approximately equal to the surface diameter $d_s$. Thus,

$$d_d \approx \sqrt{\frac{d_v^3}{d_s}}$$ (28)

Substitution for the volume diameter from Eqn. 3 and for the surface diameter from Eqns. 10, 11 and 13 leads to

$$d_a = 2R \left( \frac{2E^2}{f(E)} \right)^{1/4}$$ (29)

Relative to the volume diameter,

$$d_a = d_v \left( \frac{2E^{2/3}}{f(E)} \right)^{1/4}$$ (30)

For materials with broad size distributions it is common to combine sieving and sedimentation to provide an estimate of the complete distribution. The relationship between sedimentation and sieve sizes can be obtained from Eqns. 14, 18 and 29. Thus,

$$d_a = \frac{2d}{\sqrt{1+2E^2}} \left( \frac{E^2}{2f_0(E)} \right)^{1/4} (E \leq 1)$$ (31)

for oblate spheroids and

$$d_a = d_v \left( \frac{2E^2}{f_p(E)} \right)^{1/4} (E \geq 1)$$ (32)

for prolate.

Under laminar flow conditions, anisotropic particles such as spheroids settle in random orientation and, if the axes of the particle are inclined to the vertical, they possess a horizontal as well as a vertical component of velocity (Happel and Brenner, 1965). The effective Stokes diameter (based on the vertical component) then depends on orientation. The values given by Eqn.20 correspond to the average orientation. The orientation effect is generally small, with the effective Stokes diameter varying by about ±10% from the average (Happel and Brenner, 1965).

The effect of elongation on sedimentation size relative to volume and sieve diameters is illustrated in Fig. 4. It can be seen that sedimentation generally underestimates size relative to particle volume for both oblate and prolate spheroids. Except for oblate spheroids with elongation between 0.5 and 1.0, sedimentation methods substantially overestimate size relative to sieve size for prolate and underestimate it for highly flattened (oblate) spheroids.

3.4 Light Scattering

Particle size measurements based on light scattering estimate size from the projected area of particles in random orientation. For identical non-spherical particles this can result in an apparent size distribution due entirely to orientation. The projected area $A_p$ of a spheroidal particle oriented at an angle $\theta$ to its major axis (see Appendix) is given by

$$A_p = \pi R^2 \left( \frac{(1+E^2\tan^2\theta)(1+E^4\tan^2\theta)}{1+(1+E^4)\tan^2\theta + E^4\tan^4\theta} \right)$$ (33)

The corresponding size is the projected area diameter $d_p$ defined as the diameter of a circle with the same area as the projection of the randomly oriented particle. Thus,

$$d_p = 2\left( \frac{A_p}{\pi} \right)^{1/2}$$ (34)

For randomly oriented particles, the probability that a given particle is oriented at angle $\theta$ ($0 \leq \theta \leq \pi/2$) is equal

![Fig. 4 Effect of elongation on sedimentation size (Stokes diameter, $d_s$) relative to volume, $d_v$ and sieve, $d_A$ diameters.](image-url)
Thus, for a collection of particles, the number fraction oriented at an angle less than \( \theta \) is

\[
F_0(\theta) = \frac{2\theta}{\pi} \left(0 \leq \theta \leq \frac{\pi}{2}\right) \tag{35}
\]

For prolate spheroids, the same function expresses the fraction of randomly oriented particles whose projected area in a fixed direction is less than the value \( A_p \) as defined by Eqn. 33. In the case of oblate spheroids, the function represents the fraction with projected area greater than \( A_p \). Thus, for prolate spheroids

\[
F_0(A_p) = F_0(d_p) = F_0(\theta) \quad E \geq 1 \tag{36}
\]

while for oblate,

\[
F_0(A_p) = F_0(d_p) = 1 - F_0(\theta) \quad E \leq 1 \tag{37}
\]

3.4.1 Optical Particle Counters

Instruments based on scattering or obscuration of light by individual particles passing through a beam generally present results as a number distribution of projected area diameter. For randomly oriented identical spheroidal particles an apparent size distribution is generated according to Eqns. 33, 34, 36 and 37. Examples of such apparent distributions are given in Fig. 5. In each case, the maximum size is equal to the value of the stable projected area \( d_{ps} \) defined by Eqns. 25 and 26.

3.4.2 Laser Diffraction/Scattering

Particle sizing procedures such as the Micrtrac and Malvern systems which are based on diffraction and scattering by assemblies of suspended particles also yield apparent distributions for identical non-spherical particles due to orientation effects. These systems generally present results as volume distributions \( F_3(d_p) \). The apparent number distributions described by Eqns.33, 34, 36 and 37 can be transformed into volume fractions using

\[
F_3(\theta) = \frac{\int_0^{\pi/2} d_p^3 d\theta}{\int_0^{\pi/2} d_p^3 d\theta} \tag{38}
\]

As before, \( F_3(d_p) \) is equal to \( F_3(\theta) \) for prolate particles and \( 1 - F_3(\theta) \) for oblate. Examples of calculated volume distributions obtained by numerical integration of Eqn.38 after substitution from Eqns.33 and 34 are shown in Fig. 6.

Both the number and volume distributions show significant broadening and a general shift to larger apparent sizes due to the shape effects. In each case the effects are greatest for highly flattened particles. While the range of apparent sizes is the same for both number and volume distributions, the shift to larger apparent sizes is greater for the volume case due to the emphasis on the coarser material. In practice, these apparent distributions will be superimposed on the actual distribution of sizes.

The effects of elongation on the apparent median size

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Fig. 5  Apparent number size distributions for randomly oriented spheroids. (a) oblate \((E < 1)\), (b) prolate \((E > 1)\).

Fig. 6  Apparent volume size distribution for randomly oriented spheroids. (a) oblate, \((E < 1)\), (b) prolate, \((E > 1)\).
the number and volume distributions relative to volume diameter are compared in Fig. 7. Also included in the figure are the variations in the apparent median size relative to the sieve size and the Stokes diameter, calculated using the relationships between volume diameter and sieve size or Stokes diameter as given by Eqns. 19, 20 and 30. It can be seen that in each case the light scattering procedure overestimates size relative to the other methods for both oblate and prolate particles. The effects are generally greater for oblate particles with the exception of sieving for which overestimation is greatest for prolate particles.

3.4.3 Electrical Sensing Counters

Instruments such as the Coulter counter and Elzone systems give a direct measurement of particle volume based on the change in conductivity of an electrolyte due to displacement by suspended particles. The systems can provide the number distribution \( F_0(d_v) \) by simple counting or the volume distribution \( F_3(d_v) \) by accumulation of individual volumes. The relationships between volume diameter as determined by electrical sensing and other size measures are illustrated in Figs. 3, 4 and 7.

4. Discussion

The analysis presented above provides a simplified framework for evaluating the effects of shape on particle size estimates obtained from commonly used experimental procedures. It is interesting to evaluate the analysis using the results of experimental measurements. Comparisons of particle size data obtained by different procedures are presented in Table 1.

The elongation estimates given in the table were obtained from the expected values shown in Figs. 3, 4 and 7. The results suggest that these quartz particles were slightly flattened with an elongation of about 0.3. The quartz data indicate a relatively small deviation from the spherical shape. A more pronounced deviation can be examined using data for kaolin particles presented by Dumm (1986). In this study, median sizes of 3.75 \( \mu \)m by light scattering and 1.12 \( \mu \)m by centrifugal sedimentation were reported, giving a ratio \( d_p/d_v \) of 3.35, corresponding from Fig. 7 to an elongation of 0.063.

The volume diameter \( d_v \) can be estimated from

\[
\frac{d_v}{d_p} = \frac{d_p}{d_a} \left( \frac{d_a}{d_v} \right)
\]

(39)

Estimation using Eqn. 30, gives a value of 0.75 for the ratio \( d_a/d_v \) and of \( d_v = 1.50 \mu \)m. The average diameter \( D = 2R \) of the kaolin particles can be estimated using Eqn. 3 giving \( D = 3.77 \mu \)m. The particle thickness \( T \) is equal to \( DE \) giving \( T = 0.24 \mu \)m. These values fall well within the accepted range for kaolin particles.

Naito et al. (1998) presented a comparison of light scattering and electrical sensing data for silicon carbide whiskers. The relative apparent size was given as 1.41 which, from Fig. 7, corresponds to an elongation of either 0.3 or 10. The latter value is in good agreement with the microscopic data lengths of “several” up to about 20 \( \mu \)m and about 1 \( \mu \)m diameter reported in the paper.

It was demonstrated in Section 3.3 above that light scattering measurements on identical, non-spherical particles may result in an apparent size distribution due entirely to random orientation effects. Hogg et al. (2004)

<table>
<thead>
<tr>
<th>Methods</th>
<th>Appt. Size Ratio</th>
<th>Estimated Elongation</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sediment., Coulter</td>
<td>0.97</td>
<td>0.44 or 2.3</td>
<td>Dumm, (1986)</td>
</tr>
<tr>
<td>Sediment., Sieving</td>
<td>0.96</td>
<td>0.4</td>
<td>Cho et al., (1998)</td>
</tr>
<tr>
<td>Light Scat., Sediment.</td>
<td>1.52</td>
<td>0.28</td>
<td>Dumm, (1986)</td>
</tr>
<tr>
<td>Light Scat., Sieving</td>
<td>1.25</td>
<td>0.28</td>
<td>Austin and Shah, (1983)</td>
</tr>
<tr>
<td>Light Scat., Coulter</td>
<td>1.27</td>
<td>0.3</td>
<td>Hogg et al., 2004</td>
</tr>
<tr>
<td>Light Scat., Sieving</td>
<td>1.48</td>
<td>0.26</td>
<td>Dumm, (1986)</td>
</tr>
<tr>
<td>Coulter</td>
<td>1.11</td>
<td>0.31</td>
<td>Van Orden et al. (1979)</td>
</tr>
</tbody>
</table>
have presented light scattering results for narrow sieve fractions of glass beads and crushed quartz. The glass bead results are shown in Fig. 8, normalized for the various size fractions with respect to the particular lower sieve size $d_{AL}$.

It is clear that the size distribution by light scattering is significantly broader than the expected distribution within the sieve interval as indicated by the solid line in the figure (based on an assumed uniform distribution in that range). The broadening of the apparent distribution can probably be attributed to the somewhat limited resolution of the scattering procedure (Hogg, 2008).

Sieving errors could lead to the appearance of particles smaller than the lower sieve size but not the apparently oversize material seen in the figure. Similarly normalized results for crushed quartz are presented in Fig. 9. Despite considerable scatter among the results, it can be seen that the relative size shows no systematic variation with the actual (sieve) size, implying that the average particle shape is largely independent of size; a condition often assumed in the analysis of particle systems (Leschonski, 1984). The shift in the light scattering distribution relative to sieve size as noted previously is presumed to result from the shape effect. For sizes smaller than the median, the experimental data are in quite close agreement with a calculated, composite distribution (solid line in the figure) taking into account the distribution of sizes within the sieve fraction, assumed to be uniform (dashed line in the figure). The discrepancy in the coarser sizes is again attributed to limited resolution of the light scattering procedure, possibly compounded by deviations from the assumed smooth spheroidal shape.

5. Conclusions

The analysis presented above shows that particle shape can have significant effects on the results of experimental measurements of size. Using particle volume as a unique measure of size, it has been shown that shape effects can lead to either overestimation or underestimation of size by commonly used procedures. The spheroid model allows measured apparent size ratios obtained using different procedures, e.g., sieving/sedimentation to be used to estimate actual volume diameter and elongation and to predict results from other methods such as light scattering.

Sieving and sedimentation lead to a shift in apparent size but, for particles of similar shape, do not affect the form of the distribution. Optical (light scattering or in situ image analysis on suspended particles) and other radiation scattering procedures, on the other hand, show an apparent distribution of sizes for non-spherical particles of uniform size and shape.

Comparison of experimental apparent size of crushed quartz particles by several methods gave consistent estimates of elongation, indicating somewhat flattened particles. Comparison of apparent size of kaolin particles by light scattering and sedimentation were consistent with highly flattened particles with dimensions in broad agreement with accepted values for kaolin platelets.

Light scattering measurements on narrow sieve fractions of spherical glass beads showed no size shift but some broadening of the distribution which is attributed to limited resolution of the procedure. Similar measurements on irregular crushed quartz particles gave a significant shift in size and greater broadening of the apparent distributions, consistent with the predicted orientation effects. The results showed no systematic variation in apparent size relative to sieve size indicating that the average parti-
The analysis presented here shows that comparisons between sizing procedures can provide information on particle shape in addition to size. Rather than the common practice of presenting size as an equivalent sphere diameter, particles are characterized by volume diameter and elongation.

References


Appendix: Projected Area of Randomly Oriented Spheroids

Consider a particle with the general shape of a spheroid as shown in Fig. A1. The particle profile in the orientation shown (x-y plane) is described by the equation of an ellipse:

\[ \frac{x^2}{B^2} + \frac{y^2}{R^2} = 1 \]  
(A1)

The cross-sectional area of a transverse plane SS, drawn through the origin, at an angle \( \theta \) to the y-axis can be determined as follows: Referring to Fig. A1, the profile lies on the u-z plane perpendicular to the x-y plane and at an angle \( \theta \) to the y-z plane. The coordinates of a point on the u-axis are defined by

\[ x = u \sin \phi \]  
(A2)

\[ y = u \cos \phi \]  
(A3)

A point u, z on the profile lies on the surface of the particle and on the circumference of the circle, radius r, drawn at x in the y-z plane. The radius r can be determined using Eqn. A1 with y = r, i.e.,

\[ r^2 = R^2 \left(1 - \frac{x^2}{B^2}\right) \]  
(A4)

The vertical height of the particle profile above the point x, y is given by

\[ z^2 = r^2 - y^2 \]  
(A5)

Then, from Eqns. A2, A3 and A4,

\[ z^2 = R^2 \left(1 - \frac{u^2 \sin^2 \phi}{B^2}\right) - u^2 \cos^2 \phi \]  
(A6)

which can be expressed in the form of the equation for an ellipse in the u-z plane:

\[ \frac{z^2}{R^2} + \frac{u^2}{D^2} = 1 \]  
(A7)

with \( D \) given by

\[ \frac{1}{D^2} = \frac{\sin^2 \phi}{B^2} + \frac{\cos^2 \phi}{R^2} \]  
(A8)

The area of this elliptical profile is

\[ A_s = \pi RD \]  
(A9)

or, after substitution for D and expressing in terms of the elongation \( E \),

\[ A_s = \frac{\pi R^2 E}{\sqrt{\sin^2 \phi + E^2 \cos^2 \phi}} \]  
(A10)

![Fig. A1 Longitudinal profile of a prolate spheroid.](image-url)
An observer viewing the particle at an angle $\theta$ to the horizontal axis ($x$) sees an area defined by a plane connecting tangents to the particle surface in the viewing direction. As indicated in Fig. A1, this plane is at angle $\varphi$ to the transverse ($y$) axis of the particle. The slope of the tangent line is given by

$$\tan \theta = \frac{dy}{dx} \tag{A11}$$

Differentiation of Eqn. A1 leads to

$$\tan \theta = \frac{R^2 x}{B^2 y} \tag{A12}$$

It can be seen from Fig. A1 that

$$\tan \varphi = \frac{x}{y} \tag{A13}$$

so that

$$\tan \theta = \frac{R^2}{B^2} \tan \varphi = \frac{1}{E^2} \tan \varphi \tag{A14}$$

The projected area $A_p$ of the particle is the area of plane $SS$ projected onto the plane $PP$ perpendicular to the tangents. Thus,

$$A_p = A_s \cos \alpha \tag{A15}$$

where, from Fig. A1,

$$\alpha = \varphi - \theta \tag{A16}$$

for a prolate spheroid. The cosine can be expressed as

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} \tag{A17}$$

and

$$\tan \alpha = \tan (\varphi - \theta) = \frac{\tan \varphi - \tan \theta}{1 + \tan \varphi \tan \theta} \tag{A18}$$

Then, substituting for $\varphi$ using Eqn. A14,

$$\tan \alpha = \frac{(E^2 - 1) \tan \theta}{1 + E^2 \tan^2 \theta} \tag{A19}$$

Substitution into Eqn. A17 gives

$$\cos \alpha = \frac{1 + E^2 \tan^2 \theta}{\sqrt{1 + (1 + E^4) \tan^2 \theta + E^4 \tan^4 \theta}} \tag{A20}$$

A parallel analysis for an oblate spheroid leads to the identical result: Eqn. A20.

The area $A_s$ can also be expressed in terms of the angle $\theta$ using Eqn. A14 and the identities:

$$\sin^2 \varphi = \frac{\tan^2 \varphi}{1 + \tan^2 \varphi} \tag{A21}$$

and

$$\cos^2 \varphi = \frac{1}{1 + \tan^2 \varphi} \tag{A22}$$

Then, from Eqns. A10, A14, A15, A20, A21 and A22,

$$A_s = \pi R^2 \sqrt{\frac{(1 + E^2 \tan^2 \theta)(1 + E^4 \tan^2 \theta)}{1 + (1 + E^4) \tan^2 \theta + E^4 \tan^4 \theta}} \tag{A23}$$

Eqn. A23 describes the dependence of the projected area of a spheroid of elongation $E$ on its orientation (angle $\theta$) relative to its elliptical axis.

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**Author's short biography**

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Richard Hogg is Professor Emeritus of Mineral Processing and GeoEnvironmental Engineering at the Pennsylvania State University. He received a B.Sc. from the University of Leeds and the M.S. and PhD degrees from the University of California at Berkeley. A member of the National Academy of Engineering since 2012, Dr Hogg's research interests include fine particle processing, particle characterization, and colloid and surface chemistry.